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HOMOLOGY OF PRODUCTS AND JOINS OF REFLEXIVE RELATIONS

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The homology of products and joins of reflexive relations is computed. Rota's homology of the products of two lattices is computed. The homology of finite polyspherical posets is determined by Euler characteristic and length. The category of polyspherical posets is closed under joins and special products but not products. A special product of nonvoid reflexive relations is simply connected.

The homology of reflexive relations was introduced in [2]. All homology in this paper has constant coefficients in a principal ideal domain. An object in the category of reflexive relations is an ordered pair (X, R) where R is a reflexive relation in X. We abuse language by referring to X as a reflexive relation. The morphisms are relations preserving functions. The definitions of homotopy and excision for reflexive relations are found in [2]. Spanier's book [8] is the standard reference for singular theory.

Two points of the product $A \times B$ of reflexive relations are related iff corresponding coordinates are related (in the same direction). The topological realization functor T is not product preserving [2]. However, the product of reflexive relations is a categorical product so the Eilenberg–Zilber theorem for simplicial sets [4] together with the Künneth theorem yield the following.

Theorem 1 (Eilenberg, Zilber and Künneth). *The homology of the product $A \times B$ of reflexive relations is*

$$H_p(A \times B) \simeq [H(A) \otimes H(B)]_p + [\mathrm{Tor}(H(A), H(B))]_{p-1}\,.$$

The join $A * B$ of reflexive relations is the disjoint union of A and B. Each of A, B has its original relation and every element of A is related to every element of B. Since the topological realization functor T commutes with join and the homology of a reflexive relation is isomorphic to the singular homology of its topological realization [2], Milnor's calculation [5, p. 431] yields the following.

Theorem 2 (Milnor). *The reduced homology of the join $A * B$ of reflexive relations is*

$$\tilde{H}_{p+1}(A * B) \simeq [\tilde{H}(A) \otimes \tilde{H}(B)]_p + [\mathrm{Tor}(\tilde{H}(A), \tilde{H}(B))]_{p-1}.$$

Define $A^+ = 0 * A * 1$ and the special product $(A^+ \times B^+)^- = A^+ \times B^+ \setminus \{(0,0), (1,1)\}$ for reflexive relations A, B. This construction is of interest because Rota's homology [7] of a lattice A^+ is the homology of the poset A. In order to compute Rota's homology of the product of two lattices $A^+ \times B^+$, one computes the homology of the poset $(A^+ \times B^+)^-$ in terms of the homology of the posets A, B. (A^+ and hence $A^+ \times B^+$ is contractible.)

Lemma 3. *If one of the reflexive relations A, B is non-empty, then $(A^+ \times B^+)^-$ is path connected. $(\phi^+ \times \phi^+)^-$ is the zero sphere.*

Theorem 4. *If A, B are non-void reflexive relations, then the reduced homology of $(A^+ \times B^+)^-$ is*

$$\tilde{H}_{p+2}[(A^+ \times B^+)^-] \simeq [\tilde{H}(A) \otimes \tilde{H}(B)]_p + [\mathrm{Tor}(\tilde{H}(A), \tilde{H}(B))]_{p-1}.$$

Proof. Let $X = (A^+ \times B^+)^-$, $U_1 = X \setminus (A^+ \times 0)$, and $U_2 = X \setminus (0 \times B^+)$. $\{U_1, U_2\}$ is an excisive couple in $X = U_1 \cup U_2$. The map $f: U_1 \to U_1$ defined by $f(a, b) = (0, b)$ is a dilation retraction (see [2]) and $f[U_1]$ is isomorphic to $B * 1$ which is contractible. Since U_1, U_2 are contractible, the connecting homomorphism of the reduced Mayer–Vietoris sequence is an isomorphism. Hence

$$H_{p+2}(X) \simeq H_{p+1}(U_1 \cap U_2).$$

If $V_1 = (A * 1) \times B$ and $V_2 = A \times (B * 1)$, then $\{V_1, V_2\}$ is an excisive couple in $U_1 \cap U_2 = V_1 \cup V_2$ and $V_1 \cap V_2 = A \times B$. The inclusion of V_1 into $U_1 \cap U_2$ is homotopic to a constant map as follows: $L = \{0, 1, 2, 3\}$ is partially ordered by $0 < 1$, $1 > 2$, $2 < 3$. Pick b in B and define the homotopy $K : V_1 \times L \to U_1 \cap U_2$ by

$$K(x, y, j) = \begin{cases} (x, y), & j = 0, \\ (x, 1), & j = 1, \\ (x, b), & j = 2, \\ (1, b), & j = 3. \end{cases}$$

Since V_1 (V_2) is homotopy equivalent to $B(A)$, one may modify the reduced Mayer–Vietoris sequence of $\{V_1, V_2\}$ to obtain the exact sequence

$$0 \to \tilde{H}_{p+1}(U_1 \cap U_2) \to \tilde{H}_p(A \times B) \to \tilde{H}_p(A) + \tilde{H}_p(B) \to 0.$$

The theorem now follows as in [5, p. 431].

Let m be the set $\{0, 1, ..., m\}$. A path of length m from x to y in the reflexive relation (X, R) is a function $g : m \to X$ satisfying $g_0 = x$, $g_m = y$ and $(g_i, g_{i+1}) \in R \cup R^0$ for $0 \leqslant i < m$, where R^0 denotes the opposite of R. $P(X; x, y)$ is the set of all paths in X from x to y. E is the smallest equivalence relation in $P(X; x, y)$ containing the following U and V. $U = \{(g, g\eta^p) : g \in P(X; x, y)\}$, where η^p is a degeneracy map [4, p. 233].

If $g \in P(X; x, y)$ and either ($g_{p-1}\, R\, g_p\, R\, g_{p+1}$ and $g_{p-1}\, R\, g_{p+1}$) or ($g_{p-1}\, R^0\, g_p\, R^0\, g_{p+1}$ and $g_{p-1}\, R^0\, g_{p+1}$), then $(g, g\epsilon^p)$ is in V, where ϵ^p is a face map [4, p. 233]. In this case $0 < p <$ length g. $\pi(X, x_0)$ is the set of E-classes in $P(X; x_0, x_0)$ with multiplication defined as for the edge path group [8, p. 135]. Using [8, p. 138], $\pi(X, x_0)$ is isomorphic to the fundamental group of $(T(X), x_0)$. A similar definition of fundamental group for posets is found in [6].

Proposition 5. *If A and B are non-void reflexive relations, then the fundamental group of $(A^+ \times B^+)^-$ is singleton.*

Proof. By Lemma 3, the fundamental group of $X = (A^+ \times B^+)^-$ is independent of base point. It is sufficient to show that any loop in X based at $(0, 1)$ is equivalent to a loop in $X \setminus (A^+ \times 0)$ which is contractible. This reduces to verifying that if g is a path from g_0 to g_2 in X,

$g_0 \notin (A * 1) \times 0$ and $g_1 \in (A * 1) \times 0$, then g is equivalent to a path h from g_0 to g_3 which touches $(A * 1) \times 0$ at most at g_3. Let $g_i = (x_i, y_i)$.

If $x_1 \in A$, then set $x_p = a$ else pick $a \in A$. If $y_0 \in B$, then set $b = y_0$ else pick $b \in B$. If $y_1 \in B$, then set $b' = y_1$ else pick $b' \in B$. The path needed is

$$(x_0, y_0), (x_1, b), (a, b), (a, 1), (a, b')\,(x_1, b')\,(x_2, b')\,(x_2, y_2).$$

For elements x, y of a poset X^+ of finite length (see [2, p. 5]), $x \vee y$ is the set of all minimal upper bounds of $\{x, y\}$ in X^+ and $x \wedge y$ is defined dually. A poset X^+ of finite length is strongly upper semimodular iff $x \neq y$, and $\{x, y\}$ covers u implies that every element of $x \vee y$ covers x and y. The standard grade h for a strongly upper semimodular poset X^+ of finite length is the height function defined by Haskins and Gudder in [3, p. 360].

Proposition 6. *If X^+ is a poset of finite length, then the following are equivalent:*

(1) *X^+ is strongly upper semimodular;*

(2) *If $x, y \in X^+$, $l \in x \wedge y$, and $u \in x \vee y$, then*

$$h(x) + h(y) \geqslant h(u) + h(l).$$

Proof. Let x, y, l and u be as in (2) and assume that x and y are not comparable. If A is the set of elements in X^+ which are below u, then A is an upper semimodular poset in which the only minimal upper bound of $\{x, y\}$ is u. There is a minimal upper bound u' of $\{x, y\}$ in A for which $h(x) + h(y) \geqslant h(l) + h(u')$ (see [3, p. 368]). But $u = u'$ and the converse is clear.

Corollary 7. *If A^+, B^+ are strongly upper semimodular posets of finite length, then so is $A^+ \times B^+$.*

X is an upper polyspherical poset iff X^+ is strongly upper semimodular of finite length. The homology of a finite upper polyspherical poset is determined by its Euler characteristic and its length (called dimension in [2]). By Theorem 1, the product of two upper polyspherical posets of positive length which have non-trivial homology is not upper polyspherical.

Corollary 8. *If A and B are upper polyspherical posets, then so is $A * B$ and $(A^+ \times B^+)^-$.*

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